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Hyperspherical Harmonic Model for the Ground-State Baryons

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The wave functions of the 18 ground-state light baryons are calculated in the quark model. The space wave functions are obtained by means of a (confining) hyperspherical harmonic potential in a nonrelativistic Hamiltonian. The different flavors (up, down, strange) are treated as identical particles.

1. INTRODUCTION

In this paper we calculate the wave functions of the 18 ground-state light baryons, using the quark model with the three original flavors: up, down, and strange. The confining potential is taken to be hyperspherical harmonic, which allows for a separation between the hyperradial part and the hyperangular part of the space wave function (Sections 3.1–3.3 consider the general case of an N-body system; N is then restricted to 3 in Section 3.4). In Section 2, the flavor, baryon number, color, spin wave functions are reviewed, and in Section 3.4 the complete wave functions are defined. The employed formalism makes it possible to treat all flavors on the same footing, without the need to distinguish them on the basis of mass differences.

A remark about notation. Superscripts a, b run through the values 1, 2, 3 (the three Cartesian axes). Subscripts i, j, k run through the values $1, 2, \ldots, N$ (the N particles); subscripts u, u', u'' run through the values $1, \ldots, N-1$. The three-dimensional antisymmetric symbol is here indicated as $\varepsilon(\cdot, \cdot, \cdot)$; units are such that $\hbar = 1$; the Kronecker delta is sometimes indicated as $\delta(\cdot, \cdot)$.

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2. THE GROUND-STATE LIGHT BARYONS

A generic flavor, baryon number, color, spin state for a single quark will be designated as

$$|\mathrm{fl, bn, co, s}\rangle = |\mathrm{fl}\rangle|\mathrm{bn}\rangle|\mathrm{co}\rangle|\mathrm{s}\rangle \tag{1}$$

where (a) fl denotes the flavor eigenvalue [available flavors in our model are up (up), down (dw), strange (st); orthonormalization: $\langle \mathrm{fl}|\mathrm{fl}'\rangle = \delta(\mathrm{fl}, \mathrm{fl}')$]; (b) bn denotes the baryon number eigenvalue [available baryon numbers are $+\frac{1}{3}$ for quarks, $-\frac{1}{3}$ for antiquarks; orthonormalization: $\langle \mathrm{bn}|\mathrm{bn}'\rangle = \delta(\mathrm{bn}, \mathrm{bn}')$]; (c) co denotes the color eigenvalue [available colors are red (rd), blue (bl), green (gr); orthonormalization: $\langle \mathrm{co}|\mathrm{co}'\rangle = \delta(\mathrm{co}, \mathrm{co}')$]; (d) s denotes the z-spin eigenvalue [available values are $+\frac{1}{2}$, $-\frac{1}{2}$; orthonormalization: $\langle \mathrm{s}|\mathrm{s}'\rangle = \delta(\mathrm{s}, \mathrm{s}')$].

For any one of the 18 ground-state light baryons, we can write the flavor, baryon number, color, spin wave function in the following form:

$$|\mathrm{gb}, S\rangle = |\mathrm{Bn} = +1, \mathrm{W}\rangle \sum_F C_F(\mathrm{gb}, S)|\mathrm{fl}', \mathrm{s}'\rangle_1|\mathrm{fl}'', \mathrm{s}''\rangle_2|\mathrm{fl}''', \mathrm{s}'''\rangle_3 \tag{2}$$

where

$$F \equiv (\mathrm{fl}', \mathrm{fl}'', \mathrm{fl}''', \mathrm{s}', \mathrm{s}'', \mathrm{s}''')$$

In equation (2), gb stands for the name of the baryon (e.g., Δ^{++}, Δ^{+}, etc.) and S indicates the z-spin state ($S = +\frac{3}{2}$, $+\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$ for the decuplet baryons, $S = +\frac{1}{2}$, $-\frac{1}{2}$ for the octet baryons). The singlet of color

$$|\mathrm{Bn} = +1, \mathrm{W}\rangle = |\mathrm{bn} = +\tfrac{1}{3}\rangle_1|\mathrm{bn} = +\tfrac{1}{3}\rangle_2|\mathrm{bn} = +\tfrac{1}{3}\rangle_3$$
$$\times \frac{1}{\sqrt{6}} \sum_{\mathrm{co}', \mathrm{co}'', \mathrm{co}'''} \varepsilon(\mathrm{co}', \mathrm{co}'', \mathrm{co}''')|\mathrm{co}'\rangle_1|\mathrm{co}''\rangle_2|\mathrm{co}'''\rangle_3 \tag{3}$$

is obviously normalized, and represents a (colorless) baryonic state: $\mathrm{Bn} = +1$ is its baryon number. For antibaryons, $\mathrm{gb} \to \overline{\mathrm{gb}}$ by means of $\mathrm{bn} \to -\frac{1}{3}$ in equation (3) (and therefore $\mathrm{Bn} \to -1$). The $C_F(\mathrm{gb}, S)$ coefficients are chosen according to well-known rules, including orthonormalization and symmetry. For example,

$$|\Delta^{++}, S = +\tfrac{3}{2}\rangle = |\mathrm{Bn} = +1, \mathrm{W}\rangle\{|\mathrm{up}, +\tfrac{1}{2}\rangle_1|\mathrm{up}, +\tfrac{1}{2}\rangle_2|\mathrm{up}, +\tfrac{1}{2}\rangle_3\}$$

$$\begin{aligned}|p, S = +\tfrac{1}{2}\rangle = |\mathrm{Bn} = +1, \mathrm{W}\rangle(1/3\sqrt{2})[&2|\mathrm{up}, +\tfrac{1}{2}\rangle_1|\mathrm{up}, +\tfrac{1}{2}\rangle_2|\mathrm{dw}, -\tfrac{1}{2}\rangle_3\\ &+2|\mathrm{up}, +\tfrac{1}{2}\rangle_1|\mathrm{dw}, -\tfrac{1}{2}\rangle_2|\mathrm{up}, +\tfrac{1}{2}\rangle_3\\ &+2|\mathrm{dw}, -\tfrac{1}{2}\rangle_1|\mathrm{up}, +\tfrac{1}{2}\rangle_2|\mathrm{up}, +\tfrac{1}{2}\rangle_3\\ &-|\mathrm{up}, +\tfrac{1}{2}\rangle_1|\mathrm{up}, -\tfrac{1}{2}\rangle_2|\mathrm{dw}, +\tfrac{1}{2}\rangle_3\\ &-|\mathrm{up}, -\tfrac{1}{2}\rangle_1|\mathrm{up}, +\tfrac{1}{2}\rangle_2|\mathrm{dw}, +\tfrac{1}{2}\rangle_3\end{aligned}$$

$$-|\mathrm{up},+\tfrac{1}{2}\rangle_1|\mathrm{dw},+\tfrac{1}{2}\rangle_2|\mathrm{up},-\tfrac{1}{2}\rangle_3$$

$$-|\mathrm{up},-\tfrac{1}{2}\rangle_1|\mathrm{dw},+\tfrac{1}{2}\rangle_2|\mathrm{up},+\tfrac{1}{2}\rangle_3$$

$$-|\mathrm{dw},+\tfrac{1}{2}\rangle_1|\mathrm{up},+\tfrac{1}{2}\rangle_2|\mathrm{up},-\tfrac{1}{2}\rangle_3$$

$$-|\mathrm{dw},+\tfrac{1}{2}\rangle_1|\mathrm{up},-\tfrac{1}{2}\rangle_2|\mathrm{up},+\tfrac{1}{2}\rangle_3]$$

See, for instance, Perkins (1982) or Raspini (1984). The decuplet baryons have spin-$\frac{3}{2}$, the octet baryons have spin-$\frac{1}{2}$; experimental masses can be found in Rosner (1981) as well as in many other books and publications.

Next, we examine the space wave functions, obtained using a (confining) hyperspherical harmonic potential in a nonrelativistic Hamiltonian. This section will be more general than needed, in the sense that it applies to any N-body (nonrelativistic) hyperspherical harmonic state ($N \geq 2$). At the end, our consideration will be restricted to the ground state of the $N = 3$ case, for use with the 18 baryons. In order to simplify notation, we introduce here a compact symbolism for the $SU(6)$ flavor-spin wave functions of the aforementioned baryons. This is done by means of the definition [see equation (2)]

$$|\mathrm{gb}, S\rangle = |\mathrm{Bn} = +1, \mathrm{W}\rangle|\mathscr{U}(\mathrm{gb}, S)\rangle \tag{4}$$

3. THE HYPERSPHERICAL HARMONIC OSCILLATOR MODEL

3.1. Hamiltonian and Center-of-Mass Variables

In an inertial, Cartesian, orthogonal frame of reference $\mathscr{C}(x, y, z)$, the Hamiltonian of a nonrelativistic N-body isolated system can be written as

$$H = \sum_k \frac{\mathbf{p}_k^2}{2m_k} + V(\mathbf{r}_{ij}) \tag{5}$$

with

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j, \qquad i, j, k = 1, 2, \ldots, N$$

The symbols are standard. Here $\mathbf{r}_k = \{r_k^a\}$ and $\mathbf{p}_k = \{p_k^b\} = \{-\mathrm{i}\,\partial/\partial r_k^b\}$: $a, b = 1$, 2, 3 represent the position and canonical momentum of the particle labeled k (mass m_k), while V stands for the total internal potential.

For the identification of the "internal" Hamiltonian, we can make use of the well-known transformations (Krajcik and Foldy, 1974; Raspini, 1985):

$$\mathbf{p}_k = \boldsymbol{\pi}_k + \frac{m_k}{M}\mathbf{P}, \qquad M = \sum_k m_k \tag{6}$$

$$\mathbf{r}_k = \boldsymbol{\rho}_k + \mathbf{R} \tag{7}$$

where the internal center-of-mass (c.m.) variables $\boldsymbol{\pi}_k$, $\boldsymbol{\rho}_k$ are constrained as (Krajcik and Foldy, 1974; Raspini, 1985)

$$\sum_k \boldsymbol{\pi}_k = \mathbf{0}, \qquad \sum_k m_k \boldsymbol{\rho}_k = \mathbf{0} \tag{8}$$

The total c.m. momentum $\mathbf{P} = \sum_k \mathbf{p}_k$ and the total c.m. position $\mathbf{R} = (\sum_k m_k \mathbf{r}_k)/M$ constitute a canonically conjugated pair:

$$P^a = -\mathrm{i}\, \partial/\partial R^a \tag{9}$$

while the other nonvanishing commutators among the c.m. variables are (Krajcik and Foldy, 1974; Raspini, 1985)

$$[\rho_j^a, \pi_k^b] = \mathrm{i}\, \delta^{ab}(\delta_{jk} - m_k/M) \tag{10}$$

By means of the replacements (6)–(8), the internal Hamiltonian becomes in the standard form

$$h = \sum_k \frac{\boldsymbol{\pi}_k^2}{2m_k} + V(\boldsymbol{\rho}_{ij}) \tag{11}$$

with

$$\boldsymbol{\rho}_{ij} = \boldsymbol{\rho}_i - \boldsymbol{\rho}_j \qquad (\boldsymbol{\rho}_{ij} = \mathbf{r}_{ij})$$

The complete H operator is then given by (Krajcik and Foldy, 1974)

$$H = \mathbf{P}^2/2M + h \tag{12}$$

The space of the $3N$ coordinates r_k^a may be conveniently spanned by the set of variables R^a, η_u^b, where $\boldsymbol{\eta}_u$ are as follows:

$$\boldsymbol{\eta}_u = \boldsymbol{\rho}_{uN}, \qquad u = 1, \ldots, N-1 \tag{13}$$

The $\boldsymbol{\eta}_u$ coordinates have the property of being canonically conjugated to the $\boldsymbol{\pi}_u$ momenta, that is,

$$[\eta_u^a, \pi_{u'}^b] = \mathrm{i}\, \delta^{ab}\delta_{uu'} \tag{14}$$

which implies

$$\pi_u^a = -\mathrm{i}\, \partial/\partial \eta_u^a \tag{15}$$

Clearly, equations (9), (15), and the $\boldsymbol{\pi}$ constraint in (8),

$$\pi_N^a = \mathrm{i} \sum_u \frac{\partial}{\partial \eta_u^a} \tag{16}$$

make up a consistent set of prescriptions to express all of the momenta in (12). Furthermore, using equation (13) and the $\boldsymbol{\rho}$ constraint in (8), we can get the $\boldsymbol{\rho}$ prescriptions

$$\boldsymbol{\rho}_u = \boldsymbol{\eta}_u - \frac{1}{M} \sum_{u'} m_{u'} \boldsymbol{\eta}_{u'} \tag{17}$$

$$\boldsymbol{\rho}_N = -\frac{1}{M} \sum_{u} m_u \boldsymbol{\eta}_u \tag{18}$$

Finally, equation (12) may be written as (Raspini, 1984)

$$H = -\left(\frac{1}{2M} \nabla^2\right) - \left(\sum_{u',u''} S_{u'u''} \Delta_{u'u''}\right) + V^{\#}(\boldsymbol{\eta}_u) \tag{19}$$

where

$$\nabla^2 = \sum_a \frac{\partial^2}{\partial R^a \, \partial R^a} \tag{20}$$

$$S_{u'u''} = \left(\frac{1}{2m_{u'}} \delta_{u'u''} + \frac{1}{2m_N}\right) \tag{21}$$

$$\Delta_{u'u''} = \sum_a \frac{\partial^2}{\partial \eta^a_{u'} \, \partial \eta^a_{u''}} \tag{22}$$

$$V^{\#}(\boldsymbol{\eta}_u) = V(\boldsymbol{\rho}_{ij}(\boldsymbol{\eta}_u)) \tag{23}$$

The volume element

$$dv = \prod_k d\mathbf{r}_k \tag{24}$$

can be expressed by

$$dv = d\mathbf{R}\left(\prod_u d\boldsymbol{\eta}_u\right) \tag{25}$$

where it is convenient to set

$$\int d\mathbf{R} = D^3, \qquad D \to +\infty \tag{26}$$

3.2. Jacobi Coordinates and Hyperspherical Variables

For any chosen positive constant α, we can introduce appropriate Jacobi coordinates

$$\boldsymbol{\xi}_u = \sum_{u'} A_{uu'} \boldsymbol{\eta}_{u'} \tag{27}$$

such that the relative kinetic energy

$$\sum_k \frac{\boldsymbol{\pi}_k^2}{2m_k} = -\sum_{u',u''} S_{u'u''}\Delta_{u'u''} \tag{28}$$

is diagonalized in the following manner[2] (Ballot and Fabre De La Ripelle, 1980):

$$-\sum_{u',u''} S_{u'u''}\Delta_{u'u''} = -\frac{1}{2\alpha}\sum_u \nabla_u^2 \tag{29}$$

$$\nabla_u^2 = \sum_a \frac{\partial^2}{\partial\xi_u^a\,\partial\xi_u^a} \tag{30}$$

The H operator is then

$$H = -\left(\frac{1}{2M}\nabla^2\right) - \left(\frac{1}{2\alpha}\sum_{u'}\nabla_{u'}^2\right) + U(\boldsymbol{\xi}_u) \tag{31}$$

with

$$U(\boldsymbol{\xi}_u) = V^{\#}(\boldsymbol{\eta}(\boldsymbol{\xi}_u)) \tag{32}$$

and the volume element (24), (25) may be expressed by

$$dv = d\mathbf{R}\left(\frac{1}{|A|^3}\prod_u d\boldsymbol{\xi}_u\right) \tag{33}$$

$$|A|^3 = |\det(A_{u'u''})|^3 = [(2\alpha)^{N-1}\det(S_{u'u''})]^{-3/2} \tag{34}$$

Among the "natural" choices for α, we mention $\alpha = M$ (total mass), and $\alpha = \mu$ (reduced mass):

$$\mu = \tfrac{1}{2}[\det(S_{u'u''})]^{1/(1-N)} \tag{35}$$

The latter specification forces the Jacobi transformation to have a unitary determinant. Observe, for the calculation of μ,

$$\det(S_{u'u''}) = \frac{M}{2^{N-1}(\prod_k m_k)} \tag{36}$$

The hyperspherical variables are related to the Jacobi coordinates by the formulas (Ballot and Fabre De La Ripelle, 1980):

$$\frac{\boldsymbol{\xi}_u}{|\boldsymbol{\xi}_u|} = (\sin\theta_u\cos\phi_u)\hat{\mathbf{x}} + (\sin\theta_u\sin\phi_u)\hat{\mathbf{y}} + (\cos\theta_u)\hat{\mathbf{z}} \tag{37}$$

$$|\boldsymbol{\xi}_u| = \xi\cos\zeta_u\left(\prod_{\omega=u+1}^{\omega=N}\sin\zeta_\omega\right) \tag{38}$$

[2]For a chosen α, A is not unique. If $\check{A}$ is such that equation (29) is obtained, we have that $A = B\check{A}$ also gives (29) for the same α, provided B is an orthogonal matrix. On the other hand, the hyperradius ξ of equation (41) is the same for all the A matrices that give (29) for a fixed α.

where $\zeta_N \equiv \pi/2$, and

$$\theta_u \in [0, \pi]; \qquad \phi_u \in [0, 2\pi] \tag{39}$$

$$\zeta_u \in [0, \pi/2] \quad \text{with } \zeta_1 \equiv 0; \qquad \xi \in [0, +\infty] \tag{40}$$

The hyperradius ξ is given by (Ballot and Fabre De La Ripelle, 1980)

$$\xi = \left(\sum_u \boldsymbol{\xi}_u^2\right)^{1/2} \tag{41}$$

while equation (31) acquires the standard form (Ballot and Fabre De La Ripelle, 1980)

$$H = -\left(\frac{1}{2M}\nabla^2\right) - \left\{\frac{1}{2\alpha}\left[\frac{\partial^2}{\partial\xi^2} + \frac{3N-4}{\xi}\frac{\partial}{\partial\xi} + \frac{L^2(\Omega)}{\xi^2}\right]\right\} + U^{\#}(\xi, \Omega) \tag{42}$$

in which Ω are the hyperangular variables

$$\begin{aligned} \Omega &= \{\Omega_\lambda : \lambda = 1, \ldots, 3N-4\} \\ &= \{\zeta_t, \theta_u, \phi_u : t = 2, \ldots, N-1;\ u = 1, \ldots, N-1\} \end{aligned} \tag{43}$$

and $U^{\#}$ is the potential expressed in terms of ξ and Ω

$$U^{\#}(\xi, \Omega) = U(\boldsymbol{\xi}_u(\xi, \Omega)) \tag{44}$$

The operator $L^2(\Omega)$ is called the "grand angular momentum" and acts upon the Ω variables only. Its eigenfunctions in the Ω space are conveniently labeled $Y_L(\Omega)$ (the "hyperspherical harmonics"), where L is a group of $3N-4$ appropriate quantum numbers. A suitable (integer nonnegative) linear combination $l(L)$ of these quantum numbers yields the relationship[3]

$$L^2(\Omega)\, Y_L(\Omega) = -l(L)[l(L) + 3N - 5]\, Y_L(\Omega) \tag{45}$$

as the relevant eigenvalue equation (Ballot and Fabre De La Ripelle, 1980).

The volume element (33) can be rewritten through the usual procedure

$$dv = d\mathbf{R}\left(\frac{1}{|A|^3}\,|J(\xi, \Omega)|\, d\xi \prod_\lambda d\Omega_\lambda\right) \tag{46}$$

with

$$J(\xi, \Omega) = \det[\partial(\boldsymbol{\xi}_u)/\partial(\xi, \Omega)] \tag{47}$$

Noticing the property

$$|J(\xi, \Omega)| = \xi^{3N-4} F(\Omega) \tag{48}$$

[3]Obviously, more than one L group yield the same value of l, except for the case $l(L) = 0$, which is satisfied by just one ensemble L_0 (and Y_{L_0} is a constant).

we obtain

$$dv = d\mathbf{R}\left(\frac{1}{|A|^3}\,\xi^{3N-4}\,d\xi\right)\left[F(\Omega)\prod_{\lambda} d\Omega_\lambda\right] \tag{49}$$

and it is convenient to set

$$F(\Omega)\prod_{\lambda} d\Omega_\lambda = dv(\Omega); \qquad \frac{1}{|A|^3}\,\xi^{3N-4}\,d\xi = dv(\xi) \tag{50}$$

The usual normalization of the hyperspherical harmonics yields

$$\int dv(\Omega)\; Y_L^*(\Omega)\,Y_{L'}(\Omega) = \delta_{LL'} \tag{51}$$

3.3. The Hyperspherical Harmonic Oscillator

Next, we examine the eigenvalue problem

$$H\Psi(\mathbf{R}, \xi, \Omega) = E\Psi(\mathbf{R}, \xi, \Omega) \tag{52}$$

where H is the N-body Hamiltonian [as expressed in equation (42)] and E is the energy eigenvalue. The behavior of the center of mass is clearly separated, so that we can consider solutions

$$\Psi_{\boldsymbol{p}} = \frac{1}{D^{3/2}}\exp(i\boldsymbol{p}\cdot\mathbf{R})\Phi(\xi, \Omega) \tag{53}$$

($\boldsymbol{p}$ represents the observed eigenvalue of $\mathbf{P}$). Furthermore, if $U^{\#}$ is hypercentral,

$$U^{\#}(\xi, \Omega) = g(\xi) \tag{54}$$

one is allowed to specify the $3N-4$ hyperangular quantum numbers:

$$\Phi_L = Y_L(\Omega)R(\xi) \tag{55}$$

This leads to the equation

$$\left\{\frac{d^2R}{d\xi^2} + \frac{3N-4}{\xi}\frac{dR}{d\xi} - \frac{l(L)[l(L)+3N-5]}{\xi^2}R\right\} = 2\alpha[g(\xi)-\varepsilon]R \tag{56}$$

with

$$\varepsilon = E - \boldsymbol{p}^2/2M$$

For a hyperspherical harmonic oscillator

$$g(\xi) = \tfrac{1}{2}K\xi^2, \qquad K > 0 \tag{57}$$

The regular eigensolutions of (56) with the harmonic potential (57) are well known (Fabre De La Ripelle and Navarro, 1979; Fabre De La Ripelle, 1984), corresponding to the quantization rules

$$\varepsilon_n = \left(\frac{K}{\alpha}\right)^{1/2} \frac{2n+3(N-1)}{2}, \qquad n = 0, 1, \ldots \tag{58}$$

$$l(L) \text{ even:} \quad 0 \leq l(L) \leq n \qquad \text{if } n \text{ even} \tag{59}$$

$$l(L) \text{ odd:} \quad 1 \leq l(L) \leq n \qquad \text{if } n \text{ odd} \tag{60}$$

The $R_{nl(L)}(\xi)$ functions have the structure

$$R_{nl}(\xi) = B_{nl} \exp(-w^2/2) w^l Q_{nl}(w), \qquad w = (\alpha K)^{1/4} \xi \tag{61}$$

where $Q_{nl}(w)$ are polynomials of even powers of w, and B_{nl} are normalization constants:

$$\int dv(\xi)\, R^*_{nl}(\xi) R_{n'l}(\xi) = \delta_{nn'} \tag{62}$$

Finally we can write

$$\Psi_{pnL} = \left[\frac{1}{D^{3/2}} \exp(i\mathbf{p} \cdot \mathbf{R})\right] [Y_L(\Omega)][R_{nl(L)}(\xi)] \tag{63}$$

in which the allowed L groups are constrained by (59) and (60).

In concluding this subsection, it is worthwhile mentioning that the Hamiltonian (31), with the potential (57), is also separable in the Jacobi coordinates ξ^a_u. Therefore, an alternative complete set of regular eigensolutions (in terms of Hermite polynomials) is also handy.

3.4. The Three-Body Case; Baryons

The case $N = 2$ is clearly trivial. The case $N = 3$ is simple to treat, but still worth some explicit evaluations. If we start from the $\boldsymbol{\eta}$ variables [equation (13)] and wish to obtain the $\boldsymbol{\xi}$ variables [equation (27)], we can choose ($\alpha = M$, from now on)

$$\boldsymbol{\xi}_1 = -\left[\frac{m_1 m_2}{M(m_1+m_2)}\right]^{1/2} (\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) \tag{64}$$

$$\boldsymbol{\xi}_2 = -\left(\frac{m_3}{m_1+m_2}\right)^{1/2} \left(\frac{m_1}{M}\boldsymbol{\eta}_1 + \frac{m_2}{M}\boldsymbol{\eta}_2\right) \tag{65}$$

from which [equation (34)]

$$|A|^3 = \left(\frac{M^3}{m_1 m_2 m_3}\right)^{-3/2} \tag{66}$$

On the other hand, the relations between the $\boldsymbol{\rho}$ and the $\boldsymbol{\eta}$ variables are [equations (17), (18)]

$$\boldsymbol{\rho}_1 = -(1/M)[m_2\boldsymbol{\eta}_2 - (m_2+m_3)\boldsymbol{\eta}_1] \tag{67}$$

$$\boldsymbol{\rho}_2 = -(1/M)[m_1\boldsymbol{\eta}_1 - (m_1+m_3)\boldsymbol{\eta}_2] \tag{68}$$

$$\boldsymbol{\rho}_3 = -(1/M)(m_1\boldsymbol{\eta}_1 + m_2\boldsymbol{\eta}_2) \tag{69}$$

The hyperspherical transformation (38) can be written as ($\zeta_2 = \zeta$)

$$|\boldsymbol{\xi}_1| = \xi \sin\zeta \tag{70}$$

$$|\boldsymbol{\xi}_2| = \xi \cos\zeta \tag{71}$$

and therefore [equation (50)]

$$dv(\xi) = \left(\frac{M^3}{m_1 m_2 m_3}\right)^{3/2} \xi^5\, d\xi \tag{72}$$

$$dv(\Omega) = \sin^2\zeta \cos^2\zeta \sin\theta_1 \sin\theta_2\, d\zeta\, d\theta_1\, d\theta_2\, d\phi_1\, d\phi_2 \tag{73}$$

The hyperradius ξ is then calculated as follows [equations (41), (64), (65)]:

$$\xi^2 = \frac{1}{M^2}[m_1(m_2+m_3)\boldsymbol{\eta}_1^2 + m_2(m_1+m_3)\boldsymbol{\eta}_2^2 - 2m_1m_2\boldsymbol{\eta}_1\cdot\boldsymbol{\eta}_2] \tag{74}$$

or [equations (67)-(69)]:

$$\xi^2 = \frac{1}{M^2}[m_1m_2(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2 + m_1m_3(\boldsymbol{\rho}_1-\boldsymbol{\rho}_3)^2 + m_2m_3(\boldsymbol{\rho}_2-\boldsymbol{\rho}_3)^2] \tag{75}$$

Finally, the ground state of (63) (which is also obtainable by means of a separation in the Jacobi coordinates ξ_u^a) is given by

$$\Psi_{pnL}\big|_{n=0,l=0} = \left[\frac{1}{D^{3/2}}\exp(i\boldsymbol{p}\cdot\mathbf{R})\right]\frac{1}{\pi^{3/2}}$$

$$\times\left\{\left(\frac{Km_1m_2m_3}{M^2}\right)^{3/4}\exp[-\tfrac{1}{2}(MK)^{1/2}\xi^2]\right\} \tag{76}$$

$$\varepsilon_0 = 3(K/M)^{1/2} \tag{77}$$

If we assume that the quarks in the 18 baryons are confined by a hyperspherical harmonic potential, we can make use of the above descriptions, at least in the nonrelativistic approximation (and neglecting short-range potentials). This allows to write the complete wave function in the following way:

$$|\Psi_{\mathrm{cm}}(\mathrm{gb}, S)\rangle = [\mathrm{space}]|\mathrm{gb}, S\rangle = |\mathrm{Bn} = +1, \mathrm{W}\rangle$$

$$\times\left\{\left(\frac{Km_1m_2m_3}{\pi^2M^2D^2}\right)^{3/4}\exp[-\tfrac{1}{2}(MK)^{1/2}\xi^2]\right\}|\mathcal{U}(\mathrm{gb}, S)\rangle \tag{78}$$

Here, $\boldsymbol{p} = \mathbf{0}$ has been set, for simplicity (center-of-mass frame). Due to the nature of $|\mathcal{U}(\text{gb}, S)\rangle$, which may contain different flavors, the space wave function now operates on the $SU(6)$ wave function according to the rule

$$m_i|\text{fl}\rangle_i = m(\text{fl})|\text{fl}\rangle_i \tag{79}$$

where $m(\text{fl})$ is the fl "eigenvalue" of the m_i mass "operator" (that is, the mass of the quark in $|\cdot\rangle_i$). Note that the mass operators are clearly contained in the variables $\boldsymbol{\xi}_u$, ξ, and Ω as defined in terms of the position coordinates $\mathbf{r}_i$. (The $\boldsymbol{\eta}_u$ variables are free of the masses, and the $\mathbf{R}$ variable contains masses and positions in a symmetric combination.)

The developed formalism makes it possible to treat all flavors on the same footing. The employed Hamiltonian, in terms of the constituent variables, is expressed by (see Sections 3.1–3.3)

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + \frac{K}{4M^2} \sum_{i,j} m_i m_j (\mathbf{r}_i - \mathbf{r}_j)^2 \tag{80}$$

which is manifestly symmetric in any interchange $i \leftrightarrow j$. The potential is flavor-dependent: for an up–up–st baryon, the up–up effective elastic constant is roughly 70% of the up–st coupling [assuming usual values of the quark masses, as in Rosner (1981)].

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